

Conformal dual of a quadruplet of points

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Abstract

We define a “dual” of a quadruplet of points in S^3 in a conformal geometric way. We show that the dual of a dual quadruplet coincides with the original one. We also show that the cross ratio of the dual quadruplet is equal to the complex conjugate of that of the original one.

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1 The non-cocircular case

We start with the non-cocircular case. The cocircular case will be studied in section 2.

1.1 Definition of a dual quadruplet

Let P_1, P_2, P_3 , and P_4 be four points in S^3 which are not cocircular, and Σ a sphere through them. Let Γ_{ijk} ($i \neq j \neq k \neq i$) be an oriented circle through P_i, P_j , and P_k whose orientation is given by the cyclic order of P_i, P_j , and P_k along the circle.

Definition 1.1 Suppose $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Let \mathcal{S}_{ij} be the set of the oriented circles through P_i and P_j which intersect Γ_{ijk} and Γ_{ijl} in the same angle. Let Γ_{ij} be the circle in \mathcal{S}_{ij} which minimizes the intersection angles with Γ_{ijk} and Γ_{ijl} . We call Γ_{ij} the *circular angle bisector* of Γ_{ijk} and Γ_{ijl} .

Let $\pi_i : \Sigma \rightarrow \Pi_i \cup \{\infty\}$ ($i = 1, 2, 3, 4$) be a stereographic projection from P_i . It maps three circles through $P_i, \Gamma_{ijk}, \Gamma_{ijl}$, and Γ_{ikl} , ($\{i, j, k, l\} = \{1, 2, 3, 4\}$) to three lines which forms the triangle $\triangle \pi_i(P_j)\pi_i(P_k)\pi_i(P_l)$. Therefore, π_i maps the circular angle bisector Γ_{ij} to the angle-bisector of the angle $\angle \pi_i(P_j)$. It follows that Γ_{ij} belongs to the sphere Σ .

Three angle-bisectors $\pi_i(\Gamma_{ij}), \pi_i(\Gamma_{ik}),$ and $\pi_i(\Gamma_{il})$ meet at the incenter of the triangle $\triangle \pi_i(P_j)\pi_i(P_k)\pi_i(P_l)$. Let us denote it by \tilde{P}'_i (Figure 1).

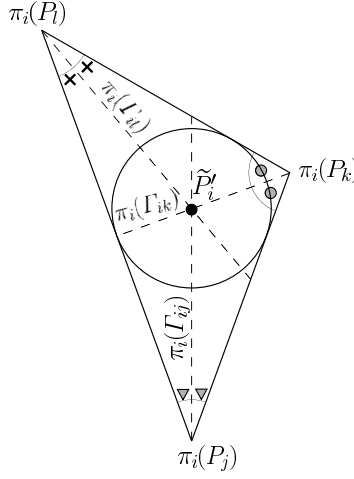


Figure 1: The inceter of the triangle $\triangle \pi_i(P_j)\pi_i(P_k)\pi_i(P_l)$

Definition 1.2 As above, put $P'_i = \pi_i^{-1}(\tilde{P}'_i)$. Namely, P'_i is given by

$$\Gamma_{ij} \cap \Gamma_{ik} \cap \Gamma_{il} = \{P_i, P'_i\} \quad (\{i, j, k, l\} = \{1, 2, 3, 4\}).$$

We call (P'_1, P'_2, P'_3, P'_4) the *dual (quadruplet)* of (P_1, P_2, P_3, P_4) .

Let us denote a quadruplet and its dual by \mathcal{Q} and \mathcal{Q}' in what follows.

It follows directly from the definition that the points of the dual quadruplet are cospherical with those of the original one.

Since the dual quadruplet can be defined by circles and angles,

Lemma 1.3 *The notion of the dual quadruplet is conformally invariant. Namely, $(T(\mathcal{Q}))' = T(\mathcal{Q}')$ for a Möbius transformation T .*

It allows us to assume that P_1, P_2, P_3 , and P_4 are four points in \mathbb{R}^3 (or $\mathbb{R}^3 \cup \{\infty\}$) which are not cocircular nor collinear.

Example 1.1 (1) When the four points are vertices of a regular tetrahedron in \mathbb{R}^3 , the dual can be obtained by the symmetry in the barycenter.

(2) The dual of

$$(1, 0, 0), (0, 1, 0), (0, -1, 0), \text{ and } (0, 0, 1)$$

are

$$\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{2}, -\frac{1}{\sqrt{2}}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right), \text{ and } \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right).$$

Our main theorem is:

Theorem 1.4 *The dual of a dual quadruplet coincides with the original one: $\mathcal{Q}'' = \mathcal{Q}$.*

1.2 Proof of Theorem 1.4

Let $(P_1'', P_2'', P_3'', P_4'')$ be the dual of (P_1', P_2', P_3', P_4') . We work in the extended plane $\pi_1(\Sigma) \cong \Pi_1 \cup \{\infty\}$ in what follows. We have only to show $\pi_1(P_1'') = \infty$.

Put

$$\tilde{P}_j = \pi_1(P_j) \quad \text{and} \quad \tilde{P}'_j = \pi_1(P'_j)$$

for $j = 1, 2, 3, 4$. We remark $\tilde{P}_1 = \infty$.

Let \tilde{P}_j ($j = 2, 3, 4$) be the excenters of the triangle $\triangle \tilde{P}_2 \tilde{P}_3 \tilde{P}_4$ (Figure 2).

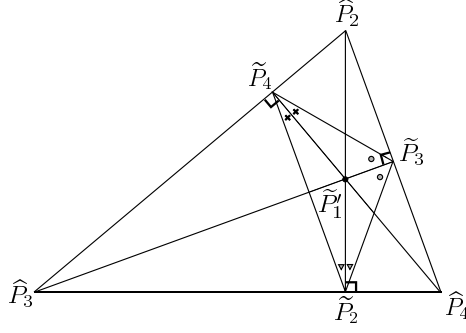


Figure 2: The incenter and excenters of $\triangle \tilde{P}_2 \tilde{P}_3 \tilde{P}_4$

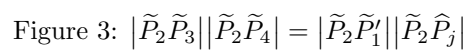
Lemma 1.5 *We have $\tilde{P}'_j = \hat{P}_j$ for $j = 2, 3, 4$. Namely, through a stereographic projection from a point of a quadruplet \mathcal{Q} , the dual can be obtained as the union of the incenter and the excenters of the triangle whose vertices are the images of the other three points of \mathcal{Q} .*

Proof. We show $\tilde{P}'_2 = \hat{P}_2$. As $\hat{P}_2, \tilde{P}_3, \tilde{P}_4$, and \tilde{P}'_1 are cocircular and $\angle \tilde{P}'_1 \tilde{P}_2 \tilde{P}_3 = \angle \tilde{P}'_1 \tilde{P}_2 \tilde{P}_4$ (Figure 3), the power of a point theorem implies

$$|\tilde{P}_2 \tilde{P}_3| |\tilde{P}_2 \tilde{P}_4| = |\tilde{P}_2 \tilde{P}'_1| |\tilde{P}_2 \hat{P}_2|. \quad (1)$$

Consider an inversion I_2 in a circle with center \tilde{P}_2 and radius $r = \sqrt{|\tilde{P}_2 \tilde{P}_3| |\tilde{P}_2 \tilde{P}_4|}$. Then

$$\begin{aligned} I_2(\tilde{P}_1) &= \tilde{P}_2, \\ \overline{\tilde{P}_2 I_2(\tilde{P}_3)} &= \overline{\tilde{P}_2 \tilde{P}_3}, & |\tilde{P}_2 I_2(\tilde{P}_3)| &= |\tilde{P}_2 \tilde{P}_4|, \\ \overline{\tilde{P}_2 I_2(\tilde{P}_4)} &= \overline{\tilde{P}_2 \tilde{P}_4}, & |\tilde{P}_2 I_2(\tilde{P}_4)| &= |\tilde{P}_2 \tilde{P}_3|, \end{aligned}$$



which implies that the triangle $\triangle I_2(\tilde{P}_1)I_2(\tilde{P}_3)I_2(\tilde{P}_4)$ is symmetric with the triangle $\triangle \tilde{P}_2\tilde{P}_3\tilde{P}_4$ in the line $\overline{\tilde{P}_2\tilde{P}_1'}$ (Figure 4 right). Since $I_2(\tilde{P}_2')$ is the incenter of the triangle $\triangle I_2(\tilde{P}_1)I_2(\tilde{P}_3)I_2(\tilde{P}_4)$, we have $I_2(\tilde{P}_2') = \tilde{P}_1'$.

On the other hand, the formula (1) implies that $I_2(\tilde{P}_2) = \tilde{P}_1'$. Therefore $\tilde{P}_2' = \tilde{P}_2$. \square

Corollary 1.6 *The dual quadruplet consists of points which are not cocircular (nor collinear when points are in \mathbb{R}^3). Therefore we can take its dual once again.*

Lemma 1.7 *Three circles through \tilde{P}_1' , $\Gamma(\tilde{P}_1', \tilde{P}_i', \tilde{P}_j')$ ($\{i, j\} \subset \{2, 3, 4\}$), have the same radius (Figure 7).*

Proof. First observe that, as $\tilde{P}_1', \tilde{P}_j', \tilde{P}_k$, and \tilde{P}_l ($\{j, k, l\} = \{2, 3, 4\}$) are cocircular (Figure 5), we have

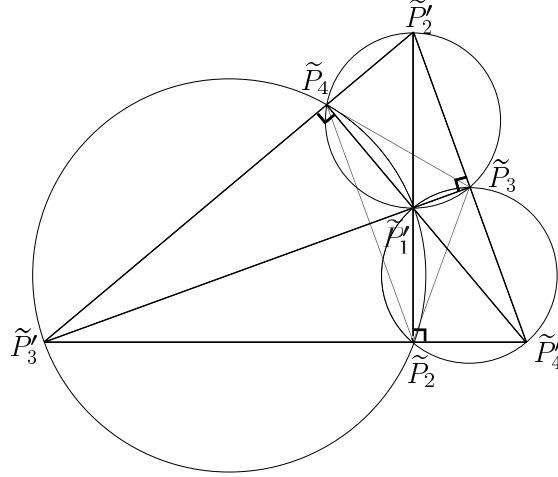


Figure 5: $\tilde{P}_1', \tilde{P}_j', \tilde{P}_k$, and \tilde{P}_l are cocircular

$$\begin{aligned}
\angle \tilde{P}_1'\tilde{P}_2'\tilde{P}_3' &= \angle \tilde{P}_1'\tilde{P}_2'\tilde{P}_4' = \angle \tilde{P}_1'\tilde{P}_3'\tilde{P}_2' = \angle \tilde{P}_1'\tilde{P}_4'\tilde{P}_2', \\
\angle \tilde{P}_1'\tilde{P}_3'\tilde{P}_2' &= \angle \tilde{P}_1'\tilde{P}_3'\tilde{P}_4' = \angle \tilde{P}_1'\tilde{P}_2'\tilde{P}_3' = \angle \tilde{P}_1'\tilde{P}_4'\tilde{P}_3', \\
\angle \tilde{P}_1'\tilde{P}_4'\tilde{P}_2' &= \angle \tilde{P}_1'\tilde{P}_4'\tilde{P}_3' = \angle \tilde{P}_1'\tilde{P}_2'\tilde{P}_4' = \angle \tilde{P}_1'\tilde{P}_3'\tilde{P}_4'
\end{aligned} \tag{2}$$

(Figure 6). The equality $\angle \tilde{P}_1'\tilde{P}_2'\tilde{P}_3' = \angle \tilde{P}_1'\tilde{P}_4'\tilde{P}_3'$ means that the inscribed angles for the chord $\tilde{P}_1'\tilde{P}_3'$ of two circles $\Gamma(\tilde{P}_1', \tilde{P}_2', \tilde{P}_3')$ and $\Gamma(\tilde{P}_1', \tilde{P}_4', \tilde{P}_3')$ coincide (Figure 7), which implies that the radii of the two circles are equal. \square

Proof of Theorem 1.4. Suppose $\mathcal{Q}'' = (P_1'', P_2'', P_3'', P_4'')$ is the dual of $\mathcal{Q}' = (P_1', P_2', P_3', P_4')$, which is the dual of $\mathcal{Q} = (P_1, P_2, P_3, P_4)$. We have only to show $\pi_1(P_1'') = \infty$.

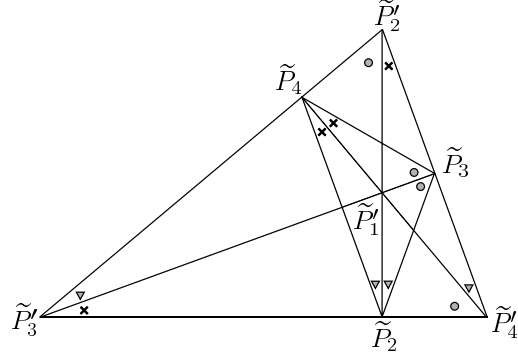


Figure 6: The equalities between angles

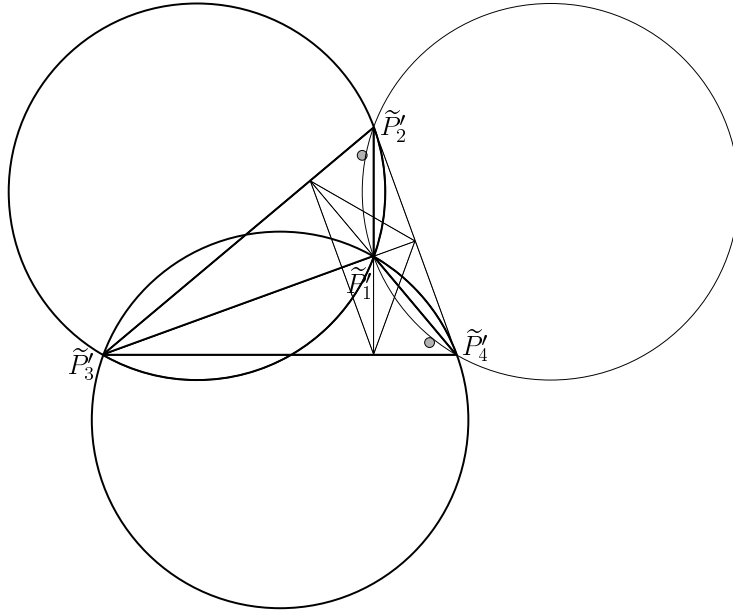


Figure 7: $\angle \tilde{P}'_1 \tilde{P}'_2 \tilde{P}'_3 = \angle \tilde{P}'_1 \tilde{P}'_4 \tilde{P}'_3 \implies r(\Gamma(\tilde{P}'_1, \tilde{P}'_2, \tilde{P}'_3)) = r(\Gamma(\tilde{P}'_1, \tilde{P}'_4, \tilde{P}'_3))$

By definition, $\pi_1(P_1'')$ belongs to the intersection of the three circular angle bisectors of pairs of circles among $\Gamma(\tilde{P}_1, \tilde{P}_i, \tilde{P}_j)$ ($\{i, j\} \subset \{2, 3, 4\}$). Since these three circles have the same radius by Lemma 1.7, any circular angle bisector of a pair among them is a straight line (Figure 8). Therefore, the three circular

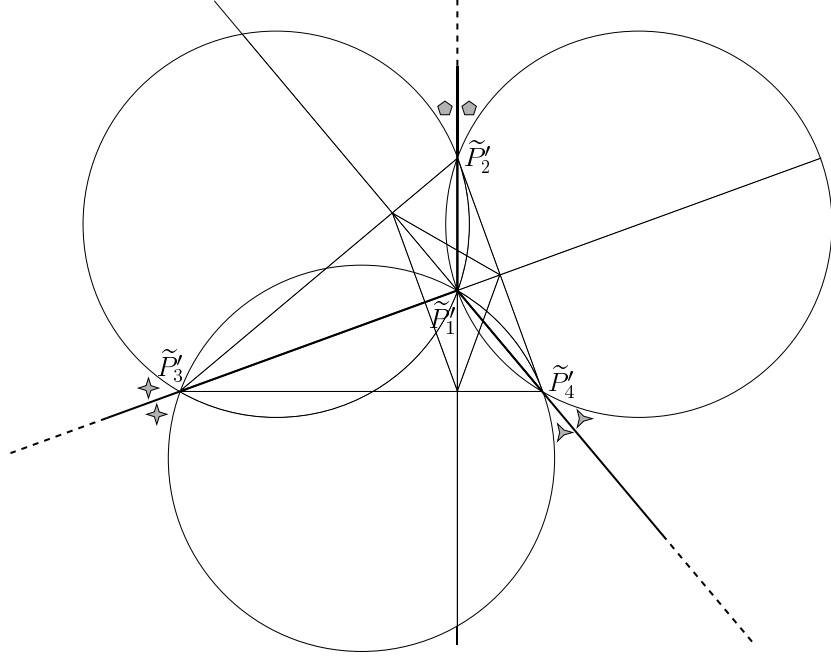


Figure 8: The three circular angle bisectors

angle bisectors meet in \tilde{P}_1' and ∞ , which completes the proof. \square

We remark we can also show $\pi_1(P_2'') = \tilde{P}_2$. Figure 5 implies

$$|\tilde{P}_2 \tilde{P}_3| |\tilde{P}_2 \tilde{P}_4| = |\tilde{P}_2 \tilde{P}_2| |\tilde{P}_2 \tilde{P}_1| = |\tilde{P}_2 \tilde{P}_4| |\tilde{P}_2 \tilde{P}_3|. \quad (3)$$

Let I' be an inversion in a circle with center \tilde{P}_2' with radius the square root of (3). Then the formula (3) implies that $I'(\tilde{P}_3') = \tilde{P}_4$, $I'(\tilde{P}_4') = \tilde{P}_3$, and $I'(\tilde{P}_1') = \tilde{P}_2$. As $I'(\pi_1(P_2''))$ is the incenter of the triangle $\triangle I'(\tilde{P}_3')I'(\tilde{P}_4')I'(\tilde{P}_1')$, we have $I'(\pi_1(P_2'')) = \tilde{P}_1'$, which implies $\pi_1(P_2'') = I'(\tilde{P}_1') = \tilde{P}_2$.

1.3 Cross ratio of the dual quadruplet

Suppose (P_1', P_2', P_3', P_4') is the dual of (P_1, P_2, P_3, P_4) , and Σ is a sphere through P_i and P_j' as before. Let p be a stereographic projection from Σ to $\mathbb{C} \cup \{\infty\}$. We identify P_i (or P_j') with the complex number $p(P_i)$ (or respectively $p(P_j')$).

Theorem 1.8 *The cross ratio of the dual quadruplet*

$$cr' = \frac{P'_2 - P'_1}{P'_2 - P'_4} \cdot \frac{P'_3 - P'_4}{P'_3 - P'_1}$$

is equal to the complex conjugate of the cross ratio of the original quadruplet

$$cr = \frac{P_2 - P_1}{P_2 - P_4} \cdot \frac{P_3 - P_4}{P_3 - P_1}.$$

Proof. First observe that the cross ratio is independent of a stereographic projection as far as it is orientation preserving. We may assume without loss of generality that $p = \pi_1$, $\tilde{P}_1 = \infty$, $\tilde{P}_4 = 0$, $\tilde{P}_2 = 1$, and $\tilde{P}_3 = cr$.

(i) Figure 6 implies

$$\arg(cr') = -\angle \tilde{P}'_1 \tilde{P}'_2 \tilde{P}'_4 - \angle \tilde{P}'_4 \tilde{P}'_3 \tilde{P}'_1 = -\angle \tilde{P}_2 \tilde{P}_4 \tilde{P}_3 = -\arg(cr).$$

(ii) Since $\tilde{P}'_1, \tilde{P}_2, \tilde{P}'_4$, and \tilde{P}_3 are cocircular, we have

$$\frac{|\tilde{P}'_2 - \tilde{P}'_1|}{|\tilde{P}'_2 - \tilde{P}'_4|} = \frac{|\tilde{P}'_2 - \tilde{P}_3|}{|\tilde{P}'_2 - \tilde{P}_2|} \quad \text{and} \quad \frac{|\tilde{P}'_3 - \tilde{P}'_4|}{|\tilde{P}'_3 - \tilde{P}'_1|} = \frac{|\tilde{P}'_3 - \tilde{P}_2|}{|\tilde{P}'_3 - \tilde{P}_2|}.$$

Since $\angle \tilde{P}'_2 \tilde{P}_3 \tilde{P}'_3 = \angle \tilde{P}'_2 \tilde{P}_2 \tilde{P}'_3 = \frac{\pi}{2}$ we have

$$|cr'| = \frac{|\tilde{P}'_2 - \tilde{P}'_1|}{|\tilde{P}'_2 - \tilde{P}'_4|} \cdot \frac{|\tilde{P}'_3 - \tilde{P}'_4|}{|\tilde{P}'_3 - \tilde{P}'_1|} = \frac{|\tilde{P}'_2 - \tilde{P}_3| \cdot |\tilde{P}'_3 - \tilde{P}_2|}{|\tilde{P}'_2 - \tilde{P}_2| \cdot |\tilde{P}'_3 - \tilde{P}_2|} = \frac{|\triangle \tilde{P}'_2 \tilde{P}'_3 \tilde{P}_3|}{|\triangle \tilde{P}'_2 \tilde{P}'_3 \tilde{P}_2|}.$$

As $\angle \tilde{P}_3 \tilde{P}_4 \tilde{P}'_2 = \angle \tilde{P}_2 \tilde{P}_4 \tilde{P}'_3$ the right hand side of the above is equal to

$$\frac{|\tilde{P}_4 - \tilde{P}_3|}{|\tilde{P}_4 - \tilde{P}_2|} = |cr|,$$

which completes the proof. \square

2 The cocircular case

We define the conformal dual of a quadruplet of cocircular points P_i by the limit of the conformal dual of a quadruplet of non-cocircular points X_i as X_i approach P_i .

Suppose $\tilde{P}_4 = 0$, $\tilde{P}_2 = 1$, $\tilde{P}_3 = c$, and $\tilde{P}_1 = \infty$ ($0 < c < 1$) after a stereographic projection from Σ to $\mathbb{C} \cup \{\infty\}$. Let $\{\tilde{X}_n\}$ be a sequence of points in $\mathbb{C} \setminus \mathbb{R}$ which approach c as n goes to ∞ . Figure 9 implies that the dual of $(\infty, 0, \tilde{X}_n, 1)$ tends to be $(c, 1, \infty, 0)$ as n goes to ∞ .

Thus we are lead to

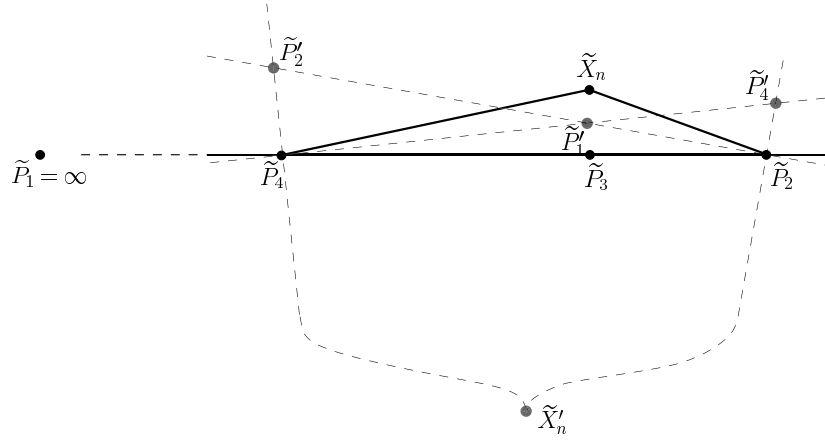


Figure 9: The four dotted lines are angle bisectors

Definition 2.1 Suppose P_1, P_2, P_3 , and P_4 are points on a circle in this order. Then the dual of (P_1, P_2, P_3, P_4) is given by (P_3, P_4, P_1, P_2) . Namely, diagonal points are being exchanged.

It is obvious that both Theorem 1.4 and Theorem 1.8 also hold for cocircular cases.

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